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Quasi-Classical Approach to Path Integrals in Supersymmetric Quantum Mechanics

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1 Introduction

Supersymmetry (SUSY) is a fascinating idea [1]. SUSY is based on the expectation that there are environments where the distinction between bosons and fermions are irrelevant. For instance, a proton and a neutron are distinct as individual particles. However, within a nucleus, they lose their identity as a proton or a neutron; they are better understood as nucleons which alternate the proton state and the neutron state, and better described by an isospin doublet. In much the same fashion, SUSY presumes a scheme in which bosons and fermions are interchangeable and are represented by a multiplet. SUSY has been playing an important role in grand unification attempts [2]. However, there has been no experimental evidence for the existence of particles necessitated by SUSY-multiplets. Although there are some indications that the SUSY scheme might work in understanding low energy phenomena [3], it is an observed fact that SUSY is generally broken. If SUSY is a physically meaningful symmetry in a certain environment, then a question arises as to how it would break down. The SUSY breaking has sometime been expected to occur at energies of the Planck mass scale ($\sim 10^{19}$ GeV). Yet, nothing rules out the possibility that SUSY breaks down at energies ($\sim$GeV) relevant to particle physics. In an effort to understand the possible breakdown at the ordinary energies, Witten [4] proposed a simple mechanism that SUSY is a good symmetry at the tree diagram level (i.e. the classical level) but spontaneously broken by small non-perturbative quantum fluctuations, and suggested to study supersymmetric quantum mechanics as a simple model. In fact, the non-perturbative breaking of SUSY in supersymmetric quantum mechanics has been well studied [5,6]. Although SUSY quantum mechanics was born in 1981 as a model to simulate dynamical breaking of SUSY in field theories, rather independently of its original intent, it has started walking alone as a legitimate method in quantum mechanics.

SUSY quantum mechanics, which is closely related to the factorization method [7,8], has been successful in a number of quantum mechanical applications. It provides exact solutions of the Schrödinger equation for a class of so-called shape invariant potentials [7, 9]. The Kaluza-Klein monopole [10], and the Dirac equation in (1+1) dimensions [11], have also been solved exactly by means of the SUSY quantization method. The d-dimensional harmonic oscillator has been shown to reveal a dynamical SUSY [12]. Even for the studies of quantum tunneling, SUSY quantum mechanics has exhibited its usefulness [13].

In recent years, the quasi-classical approximation of SUSY quantum mechanics has attracted considerable attention. In 1985, Comtet, Bandrauk and Campbell [14] proposed a quasi-classical quantization formula similar to the well-known WKB formula. This new formula, which we shall now on refer to the CBC formula, provides exact quantum spectra for all shape invariant potential systems [15]. This is in contrast to the standard WKB approach for which the Langer-like *ad hoc* modifications are necessary to obtain exact results for most of the shape invariant potentials. Even for those systems which are not shape invariant, the CBC formula seems to yield better results than WKB does [16]. Furthermore, divergent-free wave functions can be constructed within the supersymmetric quasi-classical approximation scheme [17,18].

It has been extensively discussed how good the CBC formula is. It seems unquestionable that the CBC formula is significant in quantum mechanical application. Nevertheless, there have been little discussions as to how the formula can be derived. It has been justified mainly in conjunction with the standard WKB formula. Its correctness is judged on the basis of the smallness of the difference between the CBC and the WKB formula. It is certainly desirable to find a way to derive the CBC formula as an approximation from the first principle. As the standard WKB quantization rule can be obtained from Feynman's path integral in the semi-classical approximation, it is natural to expect that the CBC formula may be found from a SUSY path integral in a similar approximation. The purpose of this paper is indeed to show how the CBC formula can be derived from a path integral proper to SUSY quantum mechanics. Although we wish to adopt the stationary phase method, we are not necessarily able to deal with real classical orbits for a supersymmetric quantum system. Therefore, we shall use the adjective "quasi-classical" rather than "semi-classical" for our stationary phase treatment.

Section 2 reviews supersymmetric quantum mechanics in a way pertinent to our discussions. In Section 3, we look into some properties of the CBC formula in comparison with the WKB formula. In Section 4, we apply a quasi-classical approach to calculate a bosonic path integral with the fermionic correction. The CBC formula is derived when SUSY is not broken. As a by-product, we obtain another WKB-like quantization formula when SUSY is broken. The quasi-classical wave functions are also obtained for the cases of good SUSY and broken SUSY.

2 SUSY Quantum Mechanics

SUSY quantum mechanics, according to Witten [4], is characterized by the *supercharge* operators Q_i $(i = 1, 2, \ldots, N)$ which are self-adjoint (i.e., $Q_i = Q_i^\dagger$) and satisfy the following graded algebra,

$$[Q_i, \mathcal{H}] = 0\,, \quad \{Q_i, Q_j\} = \mathcal{H}\delta_{ij}, \quad i = 1, 2, \ldots, N\,, \tag{2.1}$$

where $\mathcal{H}$ is the supersymmetric Hamiltonian. The algebra (2.1) implies $\mathcal{H} = 2Q_i^2$ for all i. Therefore, the Hamiltonian may as well be given in the form,

$$\mathcal{H} = \frac{2}{N}\sum_{i=1}^{N} Q_i^2. \tag{2.2}$$

Since this is positive semi-definite, spec $(\mathcal{H}) \geq 0$ in general. Let $|\,\psi\rangle$ be the eigenstate belonging to the energy eigenvalue E. Namely, $\mathcal{H}\,|\,\psi\rangle = E\,|\,\psi\rangle$. Then, $E \geq 0$. In what follows, we limit our attention to the simplest non-trivial case $N = 2$.

2.1 SUSY-Quantum Mechanics for $N = 2$

In SUSY quantum mechanics for $N = 2$, it is convenient to introduce the operators $Q = (Q_1 + iQ_2)/\sqrt{2}$ and $Q^\dagger = (Q_1 - iQ_2)/\sqrt{2}$. We see that the system obeys the algebra,

$$\mathcal{H} = \left\{Q, Q^\dagger\right\}, \quad Q^2 = 0, \quad Q^{\dagger\,2} = 0. \tag{2.3}$$

Obviously,

$$[Q, \mathcal{H}] = 0 \qquad [Q^\dagger, \mathcal{H}] = 0. \tag{2.4}$$

Let E_0 and $|\,\psi_0\rangle$ be the ground state energy and the corresponding eigenstate, respectively, so that $\mathcal{H}\,|\,\psi_0\rangle = E_0\,|\,\psi_0\rangle$. SUSY as a good symmetry for the system means

$$Q\,|\,\psi_0\rangle = Q^\dagger\,|\,\psi_0\rangle = 0\,. \tag{2.5}$$

This naturally implies $E_0 = 0$. Again, if SUSY is a good symmetry, the ground state energy must necessarily be zero.

The supersymmetry operators may be realized in terms of bosonic and fermionic operators as

$$Q = A\xi \quad Q^\dagger = A^\dagger\xi^\dagger \tag{2.6}$$

with

$$A = i\frac{1}{\sqrt{2m}}\hat{p} + \phi(\hat{q}) \quad A^\dagger = -i\frac{1}{\sqrt{2m}}\hat{p} + \phi(\hat{q}). \tag{2.7}$$

Here $\hat{q}$ and $\hat{p}$ are the usual position and momentum operators obeying $[\hat{q}, \hat{p}] = i\hbar$; ϕ is a real function of $\hat{q}$; and ξ and $\xi^\dagger$ are the Grassmann numbers having the properties,

$$\xi^2 = 0, \quad \xi^{\dagger 2} = 0, \quad \text{and} \quad \{\xi^\dagger, \xi\} = 1. \tag{2.8}$$

The supersymmetric Hamiltonian in (2.3) can now be put into the form,

$$\mathcal{H} = \frac{1}{2}\{A, A^\dagger\} + \frac{1}{2}[A, A^\dagger][\xi^\dagger, \xi], \tag{2.9}$$

or

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{1}{2}\frac{\hbar}{\sqrt{2m}}\phi'(\hat{q})[\xi^\dagger, \xi]. \tag{2.10}$$

The Grassmann numbers satisfying (2.8) may further be represented by

$$\xi = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \xi^\dagger = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

In this matrix representation, we have $[\xi^\dagger, \xi] = \sigma_3$, and hence the supersymmetric Hamiltonian is diagonalized:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{\hbar}{\sqrt{2m}}\phi'(\hat{q})\,\sigma_3, \tag{2.12}$$

or

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{2.13}$$

with

$$H_+ = A^\dagger A = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) + \frac{\hbar}{\sqrt{2m}}\phi'(\hat{q}), \tag{2.14}$$

$$H_- = AA^\dagger = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) - \frac{\hbar}{\sqrt{2m}}\phi'(\hat{q}). \tag{2.15}$$

The two Hamiltonian elements, H_+ and H_-, are often referred to as the superpartners of each other. Although the potential function $\phi(\hat{q})$ is the same as the derivative of the scalar superpotential introduced in the (0+1)-dimensional field theoretic approach to SUSY quantum mechanics [6,19], we shall follow the recent practice to call it the superpotential of SUSY quantum mechanics. The last terms of (2.14) and (2.15), which are of $\mathcal{O}(\hbar)$, stem from the presence of the fermionic variables in the charge operators (2.6). It may be viewed as a non-perturbative quantum correction originated from all fermion loops.

The eigenstates of the Hamiltonian (2.13) are doublets:

$$\psi(q) = \begin{bmatrix} \psi^{(+)}(q) \\ \psi^{(-)}(q) \end{bmatrix}. \tag{2.16}$$

Suppose that the spectrum of $\mathcal{H}$ is discrete. Then, labelling the eigenvalues and the corresponding eigenstates by the index n, we have the eigenvalue equations,

$$H_+\psi_n^{(+)} = E_n\psi_n^+ \,, \quad H_-\psi_n^{(-)} = E_n\psi_n^- \,. \tag{2.17}$$

From (2.7), (2.14) and (2.15), we get

$$\begin{aligned} A^\dagger H_+\psi_n^{(+)} &= H_- A^\dagger\psi_n^{(+)} = E_n\, A^\dagger\psi_n^{(+)}, \\ A\, H_-\psi_n^{(-)} &= H_+ A\,\psi_n^{(-)} = E_n\, A\,\psi_n^{(-)}, \end{aligned} \tag{2.18}$$

which imply

$$\psi_n^{(+)} = \frac{1}{\sqrt{E_n}}A\psi_n^{(-)} \,, \quad \psi_n^{(-)} = \frac{1}{\sqrt{E_n}}A^\dagger\psi_n^{(+)} \,, \tag{2.19}$$

for $E_n \neq 0$, and

$$0 < \mathrm{spec}(H_-) = \mathrm{spec}(H_+), \tag{2.20}$$

except for the ground state energy. Customarily, we assume that the ground state is an eigenstate of H_-.

When $E = 0$ is an eigenstate of H_-, $A^\dagger \mid \psi_0\rangle = 0$. In the q-representation, this reads

$$\left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dq} + \phi(q)\right]\psi_0(q) = 0, \tag{2.21}$$

whose solution is

$$\psi_0(q) = \psi_0(q_0)\,\exp\left[-\frac{\hbar}{\sqrt{2m}}\int_{q_0}^{q}\phi(q)\,dq\right]. \tag{2.22}$$

If SUSY is a good symmetry, this solution must exist as the ground state of the system. In other words, for a good SUSY, this solution must be normalizable (or square-integrable).

The state function (2.22) is normalizable only when the integral of the real-valued supersymmetric potential $\phi(q)$ diverges as q tends to $\pm\infty$. Thus, normalizability of $\psi_0(q)$ implies that $\phi(q)$ has an odd number of zeros [2]. If the superpotential $\phi(q)$ has a definite parity, then its parity must be odd. Otherwise, SUSY is broken. We illustrate the cases of SUSY and broken SUSY in Figure 1.

At this point, it would be interesting to notice the similarity between the supercharge operator in the matrix representation,

$$Q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix},$$

and the self-adjoint ladder operator introduced by Joseph [20]. As early as 1967 (before the idea of SUSY was put forth), Joseph found that the Hilbert space had to be doubled in order to formulate a factorization procedure with self-adjoint ladder operators. He also recognized that the algebra in the generalized space had to involve anticommutation relations.

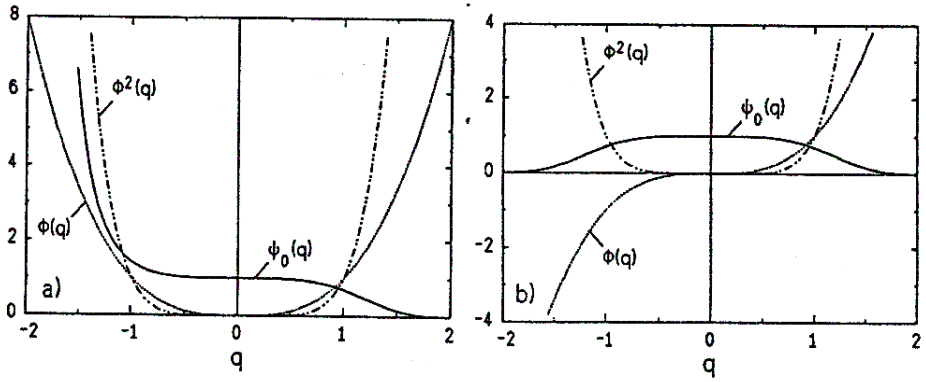

Figure 1 : SUSY quantum models for $\Phi^2(q) = q^6$. For (a), $\Phi(q) = |q|^3$ and SUSY is dynamically broken. For (b), $\Phi(q) = q^3$ and SUSY is not broken.

2.2 Remarks on the Role of Parity

Here we wish to discuss a bit more on the role of parity in SUSY quantum mechanics. At the tree level ($\hbar \to 0$), there is no distinction between H_+ and H_-. Both are reduced to

$$H_{\text{tree}} = \frac{\hat{p}^2}{2m} + \phi^2(\hat{q}) = \frac{1}{2}\left\{A, A^\dagger\right\}. \tag{2.23}$$

Suppose the superpotential $\phi(\hat{q})$ has a definite parity, that is,

$$P\phi(\hat{q}) = \pm\phi(\hat{q})\,P \tag{2.24}$$

where P is the parity operator. Evidently, the "tree" Hamiltonian, commuting with P, has even parity. When we include the quantum correction terms of $\mathcal{O}(\hbar)$, the Hamiltonian partners $H_\pm$ preserve parity, i.e.

$$[P, H_\pm] = 0, \tag{2.25}$$

only if the superpotential has odd parity; $P\phi(\hat{q}) = -\phi(\hat{q})$. In the case of odd parity, the eigenfunction (2.22) is normalizable and SUSY will be a good symmetry. On the other hand, if the superpotential has even parity, $P\phi(\hat{q}) = \phi(\hat{q})$, the Hamiltonians $H_\pm$ do not commute with the parity operator. Thus parity and SUSY are broken. In this manner, for a one-dimensional problem, violation of parity is closely linked to the dynamical breaking of SUSY. Starting with a classical one-dimensional model (i.e. a tree Hamiltonian like (2.23)) which has a definite parity,

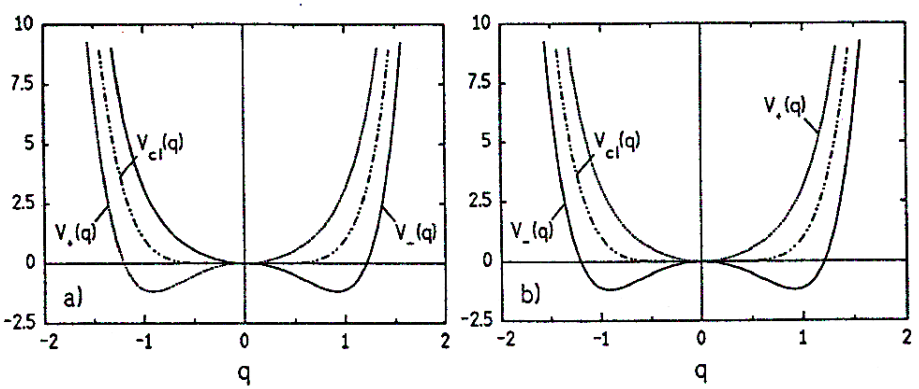

Figure 2 : The same models as in Figure 1. In (a), SUSY and parity are broken. In (b), SUSY and parity are good symmetries.

we may be led to either a model whose SUSY and parity are good symmetries or a model for which SUSY as well as parity are dynamically broken.

In Figure 2, the same model as that employed in Figure 1, i.e. $\phi^2(q) = q^6$, is used to illustrate the situation. Using units $\hbar = m = 1$ we plot the tree potential which we call the *classical potential*,

$$V_{\text{cl}}(q) = \phi^2(q) \tag{2.26}$$

and the full potentials,

$$V_\pm(q) = \phi^2(q) \pm \frac{\hbar}{\sqrt{2m}}\phi'(q). \tag{2.27}$$

The case of broken parity and broken SUSY is shown in Figure 2a. The relation $PV_\pm(q) = V_\mp(q)P$ is also demonstrated. The case where parity and SUSY are good is shown in Figure 2b.

2.3 Path Integrals for SUSY Propagators

The time evolution generated by the SUSY Hamiltonian $\mathcal{H}$ is represented by the time evolution operator,

$$\mathcal{K}(t) = e^{-(i/\hbar)\mathcal{H}t}. \tag{2.28}$$

As has been seen in Section 2, it is possible to diagonalize the Hamiltonian $\mathcal{H}$. Hence the evolution operator $\mathcal{K}$ is diagonalizable as well. In particular, on the two component fermionic basis, we have

$$\mathcal{K}(t) = \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix}, \tag{2.29}$$

where

$$K_\pm(t) = e^{-(i/\hbar)H_\pm t}. \tag{2.30}$$

In the q-representation, the components of the evolution operator yield the propagators for the two separate Hamiltonians $H_\pm$:

$$K_\pm(q'', q'; t'', t') = \langle q'' \mid e^{-(i/\hbar)H_\pm(t''-t')} \mid q' \rangle . \tag{2.31}$$

The matrix elements on the right hand side can further be reduced to the path integral form,

$$K_\pm(q'', q'; t'', t') = \int \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L_\pm(\dot{q}, q)dt\right] \mathcal{D}q\,, \tag{2.32}$$

where

$$L_\pm = \frac{1}{2}m\dot{q}^2 - \phi^2(q) \mp \frac{\hbar}{\sqrt{2m}}\phi'(q) \tag{2.33}$$

are the Lagrangians corresponding to the Hamiltonians $H_\pm$. In this way, we have obtained the proper path integrals to handle for the propagators in SUSY quantum mechanics.

There is an alternative way to formulate the path integral in SUSY quantum mechanics [5,6]. Since the Lagrangian corresponding to the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \phi^2(q) + i\xi^\dagger\dot{\xi} + \frac{\hbar}{\sqrt{2m}}\phi'(q)[\xi^\dagger, \xi], \tag{2.34}$$

it is possible to construct formally Feynmann's path integral with this Lagrangian, which naturally includes not only coordinate variables but also the Grassmann variables. After carrying out path integration over the Grassmann variables and diagonalizing the result on the two component basis, we can arrive at the same path integrals as those which we have obtained in (2.33) for the Lagrangians $L_\pm$.

3 Quasi-Classical Quantization Formulas

SUSY quantum mechanics, independently of its original intent, may be viewed as an approach to standard quantum mechanics. In principle, we can write any one-dimensional potential problem in quantum mechanics in the form,

$$H = \frac{1}{2m}p^2 + \phi^2(q) - \frac{\hbar}{\sqrt{2m}}\phi'(q) - E_0 \tag{3.1}$$

provided that the ground state energy E_0 is known [19]. In this article, since the ground state correction is a constant, we shall consider potential problems with the Hamiltonian of the form $H = H_-$ for which

$$V(q) = \phi^2(q) - \frac{\hbar}{\sqrt{2m}}\phi'(q). \tag{3.2}$$

In determining the energy spectrum of a one-dimensional system, the following quasi-classical quantization formula of Wentzel, Kramers, and Brillouin is well-known,

$$\int_a^b \sqrt{2m(E - V(q))}\,dq = \left(n + \frac{1}{2}\right)\pi\hbar\,, \tag{3.3}$$

where $q = a$ and $q = b$ are the turning points of the classical orbit, satisfying $V(a) = V(b) = E$, and $n = 0, 1, 2, \ldots$. If it is applied to the system with $H = H_-$, we have

$$\sqrt{2m}\int_a^b \left[E - \phi^2(q) + \frac{\hbar}{\sqrt{2m}}\phi'(q)\right]^{1/2} dq = \left(n + \frac{1}{2}\right)\pi\hbar\,. \tag{3.4}$$

Recently, Comtet, Bandrauk and Campbell [14] have proposed another quantization formula,

$$\int_{q_L}^{q_R} \sqrt{2m(E - \phi^2(q))}\,\mathrm{d}q = n\pi\hbar \tag{3.5}$$

where $\phi^2(q_L) = \phi^2(q_R) = E$. This formula is often referred to as the supersymmetry-inspired WKB quantization condition. Here, instead, we shall call it simply the CBC formula.

Surprisingly, the CBC formula provides exact energy spectra for the systems of so-called shape invariant potentials [7], including harmonic oscillators in one and three dimensions, the Morse oscillator, the Rosen-Morse oscillator, and the hydrogen atom among others. It is in contrast to the situation that the WKB formula (3.3) does not give exact results even for many of the shape invariant potentials unless the Langer-like *ad hoc* modification is applied together.

Although its origin is not very clear, the CBC formula has been justified in relation with the WKB approximation. The main point is as follows. If the left hand side of the WKB formula (3.4) is expanded as

$$\sqrt{2m}\int_a^b\left[E-\phi^2(q)+\frac{\hbar}{\sqrt{2m}}\phi'(q)\right]^{1/2}\mathrm{d}q$$
$$=\int_{q_L}^{q_R}\sqrt{2m(E-\phi^2(q))}\,\mathrm{d}q+\frac{\hbar}{2}\int_{q_L}^{q_R}\frac{\phi'(q)}{\sqrt{E-\phi^2(q)}}\,\mathrm{d}q+\mathcal{O}(\hbar^{3/2}),$$

the second integral on the right hand side will be equal to $\pi\hbar/2$; that is,

$$\frac{\hbar}{2}\int_{q_L}^{q_R}\frac{\phi'(q)}{\sqrt{E-\phi^2(q)}}\,\mathrm{d}q=\frac{1}{2}\pi\hbar\,. \tag{3.6}$$

Let us consider, for instance, the lowered radial harmonic oscillator with the potential

$$V(r)=\frac{1}{2}m\omega^2r^2+\frac{\ell(\ell+1)\hbar^2}{2mr^2}-\left(\ell+\frac{3}{2}\right)\hbar\omega. \tag{3.7}$$

After the Langer replacement

$$\ell(\ell+1)\;\to\;(\ell+1/2)^2, \tag{3.8}$$

the WKB formula (3.4) yields the exact spectrum $E=2n\hbar\omega$. Apparently, the ground state energy of this system is zero. The superpotential corresponding to the lowered oscillator potential (3.7) is given by

$$\phi(r)=\sqrt{\frac{m}{2}}\omega r-\frac{(\ell+1)\hbar}{\sqrt{2m}\,r}. \tag{3.9}$$

The energy spectrum calculated from the CBC formula (3.5) for this superpotential coincides with the result obtained from the WKB formula. This is expected since the relation (3.6) is easily confirmed for the superpotential (3.9). For the CBC calculation, however, the Langer replacement should not be applied. Similarly, for those whose spectra are exactly obtainable from the WKB formula with appropriate Langer-like modifications, the CBC formula provide exact results with no *ad hoc* modification. It has been observed that the WKB formula with the Langer modification can be converted into the CBC formula by the following replacements,

$$n\;\to\;n-1/2,\qquad \ell\;\to\;\ell+1/2.$$

For those which require other types of Langer-like modification, there are corresponding replacement rules for the parameters involved [21].

In the example discussed above, the harmonic oscillator potential is lowered by $\ell(\ell+1)\hbar\omega$ so as to make the resultant ground state energy vanish. Suppose SUSY is broken and the ground state energy is not zero. Since $\ell(\ell+1)=\ell'(\ell'+1)$ for $\ell'=-\ell-1$, we replace ℓ of (3.8) by $-\ell-1$ to construct the superpotential,

$$\phi(r)=\sqrt{\frac{m}{2}}\omega r+\frac{\ell\hbar}{\sqrt{2m}\,r}. \tag{3.10}$$

The corresponding potential is the one for the raised harmonic oscillator,

$$V(r)=\frac{1}{2}m\omega^2r^2+\frac{\ell(\ell+1)\hbar^2}{2mr^2}+\left(\ell-\frac{1}{2}\right)\hbar\omega. \tag{3.11}$$

The WKB formula modified with the Langer replacement (3.8) gives us the energy spectrum $E=(2n+2\ell+1)\hbar\omega$. The ground state energy is no longer zero. Since SUSY is broken, the CBC formula as given by (3.5) is not expected to yield any reasonable approximation. However, there is yet another quantization formula applicable only for the broken SUSY case [22]. It is remarkable that the Langer-modified WKB result can be reproduced, with no *ad hoc* modification, from the following formula,

$$\int_{q_L}^{q_R}\sqrt{2m(E-\phi^2(q))}\,\mathrm{d}q=\left(n+\frac{1}{2}\right)\pi\hbar \tag{3.12}$$

which is a hybrid of the WKB and the CBC formula. This formula can also produce the exact energy spectrum of the SUSY breaking Pöschl-Teller oscillator [23].

4 Quasi-Classical Path Integral Approach

In this section, we attempt to derive the CBC formula (3.4) and the new hybrid formula (3.12) by applying a method of the semi-classical approximation to the path integral (2.33) which is pertinent to SUSY quantum mechanics. The CBC formula is obtained when SUSY is good, whereas the hybrid formula follow only when SUSY is broken. To this esnd, we focus our attention mainly on a system with the Hamiltonian $H=H_-$. The Lagrangian corresponding to H_- is L_- given in (2.34).

Suppose $\lim_{q\to\pm\infty}\phi^2(q)=+\infty$, so that the energy spectrum of the system is discrete. Actually, this condition only guarantees that the spectrum of the tree

Hamiltonian (2.23) is discrete. However, as we shall see later, in a quasi-classical approximation the spectrum of H_- (and H_+) remains discrete. Since we have assumed that the ground state belongs to the system of H_-, the normalizable solution (2.22) for the vanishing ground state energy exists and SUSY is preserved only if $\phi(q) \to \infty$ as $q \to \infty$, and $\phi(q) \to -\infty$ as $q \to -\infty$. If the superpotential has the property $\lim_{q \to -\infty} \phi(q) = \infty$, then the ground state wave function is not normalizable and SUSY is broken.

We write the path integral (2.33) for H_- in the form,

$$K(x'', x'; T) = \int_{x'=q(0)}^{x''=q(T)} e^{(i/\hbar)S[q(t)]} \mathcal{D}q(t) \tag{4.1}$$

where $T = t'' - t'$ and

$$S[q(t)] = \int_0^T \left(\frac{m}{2} \dot{q}^2 - \phi^2(q) + \frac{\hbar}{\sqrt{2m}} \phi'(q) \right) \mathrm{d}t. \tag{4.2}$$

To calculate this path integral, we make a quasi-classical calculation. By the quasi-classical calculation, we mean the stationary phase approach that counts contributions only from the paths $q(t) = x(t)$ along which the the "tree" action functional,

$$S_{\text{tree}}[q(t)] = \int_0^T \left(\frac{m}{2} \dot{q}^2 - \phi^2(q) \right) \mathrm{d}t \tag{4.3}$$

is stationary. Here $\hbar \ll 1$ is assumed, and the contributions of $\mathcal{O}(\hbar)$ from the fermion loops are neglected. This approximation is based on the observation that contributions from the classical paths at the tree level will dominate in the path integral (4.1). Under the condition $\delta S_{\text{tree}}[q(t)] = 0$, we are considering an effective classical Lagrangian of the form,

$$L_{\text{cl}}(\dot{x}, x) = \frac{m}{2} \dot{x}^2 - V_{\text{cl}}(x) \tag{4.4}$$

where $V_{\text{cl}}(x) = \phi^2(x)$ is the classical potential (2.26) introduced in Section 2. The "classical" paths $x(t)$ are solutions of the Euler-Lagrange equations resulting from this Lagrangian. The constant of motion is the total energy $E = \frac{m}{2} \dot{x}^2 + V_{\text{cl}}(x)$ from which follows

$$p_{\text{cl}}(x) = m\dot{x} = \pm \sqrt{2m(E - V_{\text{cl}}(x))} \tag{4.5}$$

where the $+$ and $-$ signs signify the motions to the right and left, respectively. Note that we usually solve a classical problem for a fixed energy E.

To carry out the quasi-classical calculation, we first expand the action (4.3) about the classical path $x(t)$ by setting $q(t) = x(t) + \eta(t)$:

$$S[q(t)] \simeq S_{\text{cl}}(x'', x'; T) + \delta^2 S(T) + S_F(x'', x'; T) + \cdots \quad . \tag{4.6}$$

To second order in the "quantum fluctuations" $\eta(t)$, we have

$$S_{\text{cl}}(x'', x'; T) = \int_0^T L_{\text{cl}}(\dot{x}(t), x(t))\, \mathrm{d}t, \tag{4.7}$$

which is the classical action evaluated along the path $x(t)$, and

$$\delta^2 S(T) = \int_0^T \left(\frac{m}{2} \dot{\eta}^2 - \frac{1}{2} \left[\phi^2 \right]''(x) \eta^2 \right) \mathrm{d}t \tag{4.8}$$

which is the second order variation of (4.2). The third term in (4.6),

$$S_F(x'', x'; T) = \frac{\hbar}{\sqrt{2m}} \int_0^T \phi'(x(t))\, \mathrm{d}t, \tag{4.9}$$

stems from the fermionic quantum correction.

In order to single out clearly the dependence of $S_F(x'', x'; T)$ on $\hbar$, we introduce the quantity

$$\varphi = \frac{1}{\hbar} S_F(x'', x'; T) \tag{4.10}$$

and call it the *fermion phase*. It can be expressed as

$$\varphi = \frac{1}{\sqrt{2m}} \int_0^T \phi'(x(t))\, \mathrm{d}t = \frac{1}{2} \int_{x'}^{x''} \frac{\phi'(x)}{\pm \sqrt{E - \phi^2(x)}}\, \mathrm{d}x \tag{4.11}$$

where the last integral is to be evaluated along the classical trajectory $x(t)$.

In this approximation, the path integral (4.1), depending on (4.8), becomes Gaussian and can explicitly be calculated (see, e.g., ref. [24]), the result being

$$\begin{aligned} K(x'', x'; T) &\simeq \sqrt{\frac{\mathrm{i}}{2\pi\hbar}} \sum_{\text{fixed } T} \left[\frac{\partial^2 S_{\text{cl}}}{\partial x'' \partial x'} \right]^{1/2} \exp\left\{ \frac{\mathrm{i}}{\hbar} S_{\text{cl}} + \mathrm{i}\varphi \right\} \\ &= \sqrt{\frac{\mathrm{i}}{2\pi\hbar}} \sum_{\text{fixed } T} \left| \frac{\partial^2 S_{\text{cl}}}{\partial x'' \partial x'} \right|^{1/2} \exp\left\{ \frac{\mathrm{i}}{\hbar} S_{\text{cl}} + \mathrm{i}\varphi - \mathrm{i}\frac{\pi}{2}\mu \right\}. \end{aligned} \tag{4.12}$$

The second line indicates that the square root of the van Vleck-Pauli-Morette determinant in the sum must be carefully evaluated (principle branch). The Morse

index μ is the number of times this determinant changes its sign along the path. It is identical with the number of focal points (counted with their multiplicity) on the trajectory [24].

The sum in (4.12) is taken over all classical paths leading from x' to x'' for a *fixed time* duration T. Unfortunately, we cannot solve the classical problem for a fixed T and hence we are unable to do the sum (4.12). To circumvent this difficulty we pass over from the propagator to the energy dependent Green function

$$G(x'', x'; E) = \frac{1}{\mathrm{i}\hbar} \int_0^\infty K(x'', x'; T)\, e^{(\mathrm{i}/\hbar)ET}\, dT. \tag{4.13}$$

This integral can also be evaluated by the method of stationary phase. To this end, we shall closely follow the procedure given in Chapter 18 of Schulman's book [24] which is based on the original work of Gutzwiller [25]. The steepest descent method selects only those T values for which $E = -\partial S_{\mathrm{cl}}/\partial T$. In other words, only those paths which belong to a fixed energy E but for arbitrary $T > 0$ are taken into account. The Green function is then approximated by [24]

$$G(x'', x'; E) \simeq \frac{1}{\mathrm{i}\hbar} \sqrt{|D|} \sum_{\text{fixed } E} \exp\left\{\frac{i}{\hbar} W + \mathrm{i}\varphi - \mathrm{i}\frac{\pi}{2}\nu\right\} \tag{4.14}$$

where the sum is now taken over all classical paths $x(t)$ from x' to x'' with a *fixed energy* E. In the above $W(x'', x'; E)$ is Hamilton's characteristic function and is obtained from the classical action (4.8) through a Legendre transformation,

$$W(x'', x'; E) = S_{\mathrm{cl}}(x'', x'; T) + ET = \int_{x'}^{x''} p_{\mathrm{cl}}(x)\, \mathrm{d}x. \tag{4.15}$$

The last integral is again evaluated along the classical path $x(t)$. The determinant D appearing in (4.14) can now be explicitly calculated and given by [24]

$$|D| = \frac{m}{2} \left[(E - V_{\mathrm{cl}}(x''))(E - V_{\mathrm{cl}}(x'))\right]^{-1/2} = \frac{m^2}{|p_{\mathrm{cl}}(x'')p_{\mathrm{cl}}(x')|}. \tag{4.16}$$

This gives rise to the so-called Maslov index ν which is for one-dimensional problems identical with the number of turning points along the path [24]. In fact, as long as $\partial E/\partial T > 0$ the Maslov index ν is identical with the Morse index μ [24].

For each classical trajectory with the same fixed energy value E we have to find W, φ and ν in order to do the summation in (4.14). For this purpose, we divide the set of all paths into four classes [24]. To the first class belong those which leave

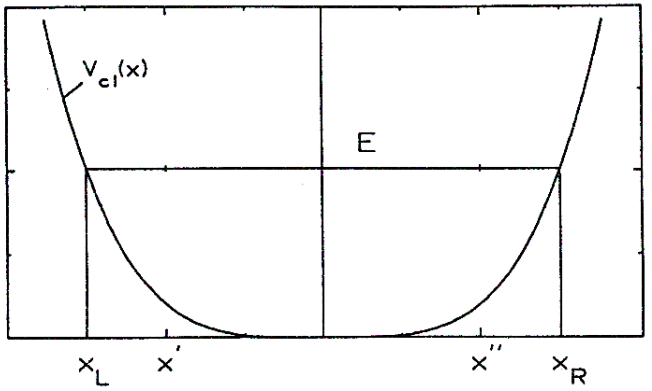

Figure 3 : An illustration for the evaluation of the path sum (4.14).

x' to the right and reach x'' from the left. This class may be characterized by the ordered pair of indices (R, L). Similarly, the remaining three classes are indexed by the ordered pairs of indices, (L, L), (R, R) and (L, R). Within each class a particular path is uniquely determined by the number of full cycles it involves. The class will be indicated by a superscript in brackets and the member of full cycles by a subscript, e.g. $\varphi_k^{(i)}$. Let us denote the classical turning points (see Figure 3 for an illustration) by x_L and x_R, respectively, and define

$$w(x) = \int_{x_L}^{x} \sqrt{2m(E - V_{\mathrm{cl}}(q))}\, \mathrm{d}q\,, \quad a(x) = \arcsin\frac{\phi(x)}{\sqrt{E}}. \tag{4.17}$$

With these we can calculate all relevant quantities appearing in the sum (4.15). For the shortest path in each class we have:

Class (R, L): This is the direct path from x' to x'' with no reflection (Figure 3).

$$\begin{aligned} W_0^{(1)} &= \int_{x'}^{x''} \sqrt{2m(E - \phi^2(x))}\, \mathrm{d}x = w(x'') - w(x') \\ \nu_0^{(1)} &= 0 \\ \varphi_0^{(1)} &= \frac{1}{2}\int_{x'}^{x''} \frac{\phi'(x)}{\sqrt{E - \phi^2(x)}}\, \mathrm{d}x = \frac{1}{2}\left[a(x'') - a(x')\right] \end{aligned} \tag{4.18}$$

Class (L, L): This path starts from x', is reflected once at x_L, and reaches x''.

$$W_0^{(2)} = w(x'') + w(x')$$

$$\begin{aligned} \nu_0^{(2)} &= 1 \\ \varphi_0^{(2)} &= \frac{1}{2}\left[a(x'') + a(x')\right] - a(x_L) \end{aligned} \tag{4.19}$$

Class (R,R): This path starts at x', passes x'' once, is reflected at x_R, and arrives at x'' again from right.

$$\begin{aligned} W_0^{(3)} &= 2w(x_R) - w(x'') - w(x') \\ \nu_0^{(3)} &= 1 \\ \varphi_0^{(3)} &= a(x_R) - \frac{1}{2}\left[a(x'') + a(x')\right] \end{aligned} \tag{4.20}$$

Class (L,R): This path, during its passage from x' to x'', is reflected first at x_L and later at x_R.

$$\begin{aligned} W_0^{(4)} &= 2w(x_R) - w(x'') + w(x') \\ \nu_0^{(4)} &= 2 \\ \varphi_0^{(4)} &= a(x_R) - a(x_L) - \frac{1}{2}\left[a(x'') - a(x')\right] \end{aligned} \tag{4.21}$$

From these results we can obtain the corresponding expressions for an arbitrary path (making k cycles, $k \in \mathrm{N}_0$) in each class:

$$\begin{aligned} W_k^{(i)} &= W_0^{(i)} + 2kw(x_R) \\ \nu_k^{(i)} &= \nu_0^{(i)} + 2k \\ \varphi_k^{(i)} &= \varphi_0^{(i)} + k\left[a(x_R) - a(x_L)\right]. \end{aligned} \tag{4.22}$$

Now we are ready to perform the summation in (4.14) at least partially. First, we rewrite this sum as

$$\begin{aligned} \sum_{\text{fixed } E} \exp\left\{\frac{\mathrm{i}}{\hbar}W + \mathrm{i}\varphi - \mathrm{i}\frac{\pi}{2}\nu\right\} &= \sum_{i=1}^{4} \exp\left\{\frac{\mathrm{i}}{\hbar}W_0^{(i)} + \mathrm{i}\varphi_0^{(i)} - \mathrm{i}\frac{\pi}{2}\nu_0^{(i)}\right\} \\ &\times \sum_{k=0}^{\infty} \exp\left\{\mathrm{i}k\left[\frac{2w(x_R)}{\hbar} + a(x_R) - a(x_L) - \pi\right]\right\}. \end{aligned} \tag{4.23}$$

Then we notice that the second sum is a geometric series which is easily computed. Therefore, in the quasi-classical approximation, the Green function can be expressed in the form,

$$G(x'',x';E) \simeq \frac{m}{\mathrm{i}\hbar\sqrt{|p_{\mathrm{cl}}(x'')p_{\mathrm{cl}}(x')|}} \frac{\sum_{i=1}^{4} \exp\left\{(\mathrm{i}/\hbar)W_0^{(i)} + \mathrm{i}\varphi_0^{(i)} - \mathrm{i}(\pi/2)\nu_0^{(i)}\right\}}{1 - \exp\left\{\mathrm{i}\left[2w(x_R)/\hbar + a(x_R) - a(x_L) - \pi\right]\right\}}. \tag{4.24}$$

From this Green function, we are able to find the energy spectrum and the energy eigenfunctions.

4.1 Quasi-Classical Quantization Conditions

The poles of the Green function (4.24) give rise to the quasi-classical energy spectrum. They occur when

$$w(x_R) \equiv \int_{x_L}^{x_R} p_{\mathrm{cl}}(x)\,\mathrm{d}x = \hbar\pi\left(n + \frac{1}{2} - \frac{a(x_R) - a(x_L)}{2\pi}\right), \quad n \in \mathrm{N}_0. \tag{4.25}$$

For the explicit values of $a(x_L)$ and $a(x_R)$, we have to go back to the definition of the classical turning points, $\phi^2(x_L) = \phi^2(x_R) = E$. These equalities are equivalent to $|\phi(x_L)| = |\phi(x_R)| = \sqrt{E}$ which have the two solutions (the other two possibilities are excluded by the assumption that $\phi(x) \to \infty$ as $x \to \infty$).

Case I: $\phi(x_R) = \phi(x_L) = \sqrt{E}$. This corresponds to a broken SUSY as has been discussed in Section 2. From (4.17), we have $a(x_R) = a(x_L) = \arcsin 1 = \pi/2$. Thus, (4.25) immediately yields

$$\int_{x_L}^{x_R} p_{\mathrm{cl}}(x)\,\mathrm{d}x = \hbar\pi\left(n + \frac{1}{2}\right), \quad n \in \mathrm{N}_0. \tag{4.26}$$

This is nothing else but the hybrid formula (3.12). Of course, this is the usual WKB result for the "tree" Hamiltonian (2.23). In other words, for broken SUSY the quasi-classical approximation is the same as the standard semi-classical WKB expression for H_{tree}. Since we are dealing with the system of $H = H_-$, the actual potential is of the form $V(q) = \phi^2(q) - (\hbar/\sqrt{2m})\phi'(q)$, the standard WKB formula if applied in quantum mechanics must be calculated with the actual potential $V(q)$. The present formula (4.26) where the derivative term is omitted from the potential apparently differs, when applied to quantum mechanics, from the standard WKB

formula [23]. Although the fermion loops do not affect the spectrum in this order of $\hbar$, as we will see in the next section there is an effect on the quasi-classical eigenfunctions.

Case II: $\phi(x_R) = -\phi(x_L) = \sqrt{E}$. In this case SUSY is a good symmetry and we have $a(x_R) = -a(x_L) = \arcsin 1 = \pi/2$. This leads us to the expression

$$\int_{x_L}^{x_R} p_{\rm cl}(x)\,\mathrm{d}x = \hbar\pi n\ , \quad n \in \mathbf{N}_0 \tag{4.27}$$

which is the celebrated formula of Comtet, Bandrauk and Campbell [14]. This formula (4.27) gives the exact ground state energy $E_0 = 0$ for $n = 0$. Here we wish to emphasize that if SUSY is a good symmetry the quasi-classical calculation of Feynman's path integral very naturally leads us to the CBC formula. The contribution of the fermion phase φ is also very clear in this approach.

The results obtained above are for the system $H = H_-$. The same calculation may be made for the Hamiltonian H_+ for which the quasi-classical quantization rules are found as follows.

Case I (broken SUSY):

$$\int_{x_L}^{x_R} p_{\rm cl}(x)\,\mathrm{d}x = \hbar\pi\left(n + \frac{1}{2}\right)\ , \quad n \in \mathbf{N}_0. \tag{4.28}$$

Case II (good SUSY):

$$\int_{x_L}^{x_R} p_{\rm cl}(x)\,\mathrm{d}x = \hbar\pi\,(n+1)\ , \quad n \in \mathbf{N}_0. \tag{4.29}$$

We see that the relations (2.17) and (2.18) between the spectra of H_+ and H_- are still valid in this approximation.

4.2 Quasi-Classical Energy Eigenfunctions

The energy eigenfunctions can be obtained from the residues of (4.24) at the poles $E = E_n$. To calculate the residues, we have to perform the remaining summation in the Green function (4.24). Again we treat the two cases of SUSY and broken SUSY separately.

Case I (broken SUSY): The remaining sum in (4.24) becomes in this case a product of two sine functions,

$$\sum_{i=1}^{4} \exp\left\{\frac{i}{\hbar}W_0^{(i)} + i\varphi_0^{(i)} - \frac{i\pi}{2}\nu_0^{(i)}\right\} = 4\sin\left[\frac{w(x')}{\hbar} + \frac{a'(x')}{2}\right]\sin\left[\frac{w(x'')}{\hbar} + \frac{a(x'')}{2}\right].$$

and hence the residues of $G(x'', x'; E)$ at the pole $E = E_n$ can easily be calculated:

$$\operatorname{Res} G(x'', x'; E)\Big|_{E=E_n} \simeq \frac{4m}{\tau_n}\,\frac{\sin\left(\frac{1}{\hbar}w(x') + \frac{1}{2}a(x')\right)}{\sqrt{|p_{\rm cl}(x')|}}\,\frac{\sin\left(\frac{1}{\hbar}w(x'') + \frac{1}{2}a(x'')\right)}{\sqrt{|p_{\rm cl}(x'')|}} \tag{4.30}$$

where

$$\tau_n = \sqrt{2m}\int_{x_L}^{x_R} \frac{1}{\sqrt{E_n - \phi^2(x)}}\,\mathrm{d}x \tag{4.31}$$

is the period of the bounded classical motion with energy E_n. From this, we obtain the wave function for a fixed eigenvalue $E = E_n > 0$ $(x_L < x < x_R)$:

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}\,|p_{\rm cl}(x)|^{-1/2}\sin\left(\frac{1}{\hbar}\int_{x_L}^{x} p_{\rm cl}(q)\,\mathrm{d}q + \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}\right)\ , \quad n \in \mathbf{N}_0, \tag{4.32}$$

which is properly normalized [26].

Case II: (good SUSY): The remaining sum in (4.24) can similarly be evaluated, which turns out to be a product of two cosine functions in this case, and the corresponding residues are given by

$$\operatorname{Res} G(x'', x'; E)\Big|_{E=E_n} = \frac{4m}{\tau_n}\,\frac{\cos\left(\frac{1}{\hbar}w(x') + \frac{1}{2}a(x')\right)}{\sqrt{|p_{\rm cl}(x')|}}\,\frac{\cos\left(\frac{1}{\hbar}w(x'') + \frac{1}{2}a(x'')\right)}{\sqrt{|p_{\rm cl}(x'')|}}. \tag{4.33}$$

From this, again, we obtain the normalized wave function ($E = E_n > 0$, i.e. $n > 0$)

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}\,|p_{\rm cl}(x)|^{-1/2}\cos\left(\frac{1}{\hbar}\int_{x_L}^{x} p_{\rm cl}(q)\,\mathrm{d}q + \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}\right)\ , \quad n \in \mathbf{N}. \tag{4.34}$$

These results may be compared with the usual WKB wave function [24] for the tree Hamiltonian (2.23),

$$\Phi_n(x) = \sqrt{\frac{4m}{\tau_n}}\,|p_{\rm cl}(x)|^{-1/2}\sin\left(\frac{1}{\hbar}\int_{x_L}^{x} p_{\rm cl}(q)\,\mathrm{d}q + \frac{\pi}{4}\right). \tag{4.35}$$

What makes our quasi-classical energy eigenfunctions different from this WKB result is the presence of the extra non-constant phase $\frac{1}{2}\arcsin\left[\phi(x)/\sqrt{E_n}\right]$ in the place of a constant phase $\pi/4$.

This phase changes in case I (broken SUSY) from $\pi/4$ to zero and back to $\pi/4$ as x varies from x_L to x_R. In case II (SUSY), it starts at the value $-\pi/4$ for x_L and ends with the value $\pi/4$ at x_R. For broken SUSY the wave function is similar to the WKB wave function (4.35) given as a "sine" function. When SUSY is good, the wave function is a "cosine" function. This is in contrast to some recent claims [17] which suggested the sine as the correct expression [17] but is in agreement with that of ref. [18].

For completeness, we give also the results for the Hamiltonian H_+:

Case I (broken SUSY):

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}\,|p_{\rm cl}(x)|^{-1/2}\cos\left(\frac{1}{\hbar}\int_{x_L}^{x} p_{\rm cl}(q)\,{\rm d}q - \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}\right)\ ,\quad n\in{\rm N}_0 \quad (4.36)$$

Case II (good SUSY):

$$\Psi_n(x) = \sqrt{\frac{4m}{\tau_n}}\,|p_{\rm cl}(x)|^{-1/2}\sin\left(\frac{1}{\hbar}\int_{x_L}^{x} p_{\rm cl}(q)\,{\rm d}q - \frac{1}{2}\arcsin\frac{\phi(x)}{\sqrt{E_n}}\right)\ ,\quad n\in{\rm N}_0. \quad (4.37)$$

In the above, we have made a quasi-classical analysis of the path integral in SUSY quantum mechanics, and derived the quantization rule of Comtet, Bandrauk and Campbell. As a by-product, we have obtained another quantization rule which is applicable only when SUSY is broken. Although the new formula reproduces exact spectra for some of the shape-invariant potential systems, it is still an open question in what extent the formula is significant.

The quasi-classical analysis we have made for the one-dimensional path integral can easily be extended to the radial path integral in three dimensions [23]. The parity discussion applicable to the one-dimensional problem has to be modified in the three-dimensional case.

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